1. (10%) Let $A = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$. Orthogonally diagonalize A.

2. (20%, 5% each) $A = \begin{bmatrix} 1 & 2 & -5 & 11 & -3 \\ 2 & 4 & -5 & 15 & 2 \\ 1 & 2 & 0 & 4 & 5 \\ 3 & 6 & -5 & 19 & -2 \end{bmatrix}$.

   a. Row reduce A to its reduced echelon form.
   b. Find a basis for Col A, where Col A is the column space of A. Also, find a basis for Row A, where Row A is the row space of A. Are rows of A linearly independent?
   c. Find a basis for Nul A, where Nul A is the null space of A.
   d. Evaluate rank(A), dim Nul A, rank(A^T), and dim Nul A^T, respectively.

3. (20%, 10% each) Let $\mathbf{y} = \begin{bmatrix} 4 & 2 & 0 \end{bmatrix}^T$ and $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$ where $\mathbf{u}_1$ and $\mathbf{u}_2$ are specified below. Suppose that $\mathbf{y} = \hat{\mathbf{y}} + z$ where $\hat{\mathbf{y}}$ is in W and $\mathbf{z}$ is in $W^\perp$ ($W^\perp$ is the orthogonal complement of W).

   a. If $\mathbf{u}_1 = \begin{bmatrix} 2 & -1 & 3 \end{bmatrix}^T$ and $\mathbf{u}_2 = \begin{bmatrix} 1 & -1 & -1 \end{bmatrix}^T$. Is $\{\mathbf{u}_1, \mathbf{u}_2\}$ an orthogonal basis for W?

      Find $\hat{\mathbf{y}}$ and $\mathbf{z}$ accordingly.

   b. If $\mathbf{u}_1 = \begin{bmatrix} 1 & 2 & 0 \end{bmatrix}^T$ and $\mathbf{u}_2 = \begin{bmatrix} 0 & 1 & -1 \end{bmatrix}^T$. Is $\{\mathbf{u}_1, \mathbf{u}_2\}$ an orthogonal basis for W?

      Find $\hat{\mathbf{y}}$ and $\mathbf{z}$ accordingly.

4. (10%, 5% each) Throw a pair of fair dice. Let X_1 be the outcome of the first die and X_2 be the outcome of the second die. Let A be the event $X_1 < 4$ and B be the event $X_1 + X_2 > 9$.
   a. Calculate $P(A)$ and $P(B)$.
   b. Are A and B independent? Why?

5. (20%) Suppose that the probability density function of a random variable X is given by

   $$f_X(x) = \begin{cases} kx, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

   a. Find the value of k. (5%)
   b. Evaluate the variance of X. (5%)
   c. If X is rounded to its nearest integer (0 or 1), find the average squared quantization error. (10%)

6. (20%) Consider a pair of random variables X and Y with joint probability density function given by

   $$f_{XY}(x,y) = \begin{cases} kxy, & 0 \le x, y \le 1 \\ 0, & \text{otherwise} \end{cases}$$

   a. Find the constant k. (5%)
   b. Are X and Y independent? Why? (5%)
   c. Let $Z = X + Y$. Find $f_Z(z)$, the probability density function of Z. (10%)