115EE05

國立臺北科技大學 115 學年度碩士班招生考試

系所組別：2140 電機工程系碩士班丁組

第一節　線性代數　試題

第 1 頁　共 1 頁

注意事項：

1. 本試題共 8 題，第 1 題，第 2 題，第 6 題和第 8 題都 10 分。第 3 題，第 4 題，第 5 題和第 7 題都 15 分，共 100 分。
2. 不必抄題，作答時請將試題題號及答案依照順序寫在答案卷上。
3. 全部答案均須在答案卷之答案欄內作答，否則不予計分。
4. 所有答案均需詳細推導或說明，否則不予計分。

1. (10 分) Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}.$$

(a) (4 分) Determine whether the columns of $\mathbf{A}$ are linearly independent.
(b) (4 分) Find a basis for the column space $C(\mathbf{A})$.
(c) (2 分) State the rank of $\mathbf{A}$.

2. (10 分) Let $\mathbf{x} = (1, -1, 2)^T$ and $\mathbf{y} = (2, 0, 1)^T$. T is a transpose operator.
(a) (2 分) Compute the dot product $\mathbf{x} \cdot \mathbf{y}$.
(b) (2 分) Determine whether $\mathbf{x}$ and $\mathbf{y}$ are orthogonal.
(c) (6 分) Find the projection of $\mathbf{x}$ onto $\mathbf{y}$.

3. (15 分) Solve the following system of linear equations using **Gaussian elimination**:

$$x + y + z = 2,$$

$$2x + 3y + z = 5,$$

$$3x + 4y + 2z = 8.$$

(a) (5 分) Write the augmented matrix.
(b) (5 分) Perform row operations to reach row echelon form.
(c) (5 分) Determine whether the system has a unique solution, infinitely many solutions, or no solution.

4. (15 分) Let

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 1 & 1 \end{bmatrix}.$$

(a) (5 分) Find a basis for the null space $N(\mathbf{A})$.
(b) (5 分) Determine the dimension of $N(\mathbf{A})$.
(c) (5 分) Verify the rank-nullity theorem for this matrix $\mathbf{A}$.

5. (15 分) Given the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Please apply the **Gram–Schmidt process** to construct an orthonormal basis for the subspace spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$.

6. (10 分) Consider the overdetermined system $\mathbf{A}\mathbf{x} = \mathbf{b}$, where

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}.$$

(a) (2 分) Write down the normal equations.
(b) (4 分) Find the least-squares solution of $\mathbf{x}$.
(c) (4 分) Briefly explain the geometric meaning of the least-squares solution.

7. (15 分) Let

$$\mathbf{A} = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$$

(a) (5 分) Find the eigenvalues of $\mathbf{A}$.
(b) (5 分) Find a corresponding eigenvector for each eigenvalue.
(c) (5 分) Determine whether $\mathbf{A}$ is diagonalizable. Justify your answer.

8. (10 分)
Let $\mathbf{A}$ be a real symmetric matrix.
(a) (5 分) State the **Spectral Theorem**.
(b) (5 分) Explain why eigenvectors corresponding to distinct eigenvalues of $\mathbf{A}$ are orthogonal.