國立臺北科技大學 112 學年度碩士班招生考試

系所組別：2142 電機工程系碩士班丁組

第一節　線性代數　試題（選考）

注意事項：

1. 本試題共 5 題，共 100 分。
2. 不必抄題，作答時請將試題題號及答案**依照順序**寫在答案卷上。
3. 全部答案均須在答案卷之答案欄內作答，否則不予計分。
4. 各題答案均須**完整推導**，否則將酌予扣分。

1. (20%) Least-squares (LS) solution:

(1) (10%) Find an LS solution of $\mathbf{Ax} = \mathbf{b}$ for

$$\mathbf{A} = \begin{bmatrix} -1 & 2 \\ 2 & -3 \\ -1 & 3 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

(2) (10%) Explain why we need the LS solution for a linear system.

2. (15%) Consider an N x N matrix $\mathbf{H}$ and a non-zero N x 1 vector $\mathbf{x}$. It is known that

$$\lambda_{\max}^2 \mathbf{x}^H \mathbf{x} \geq \mathbf{x}^H \mathbf{H}^H \mathbf{H} \mathbf{x} \geq \lambda_{\min}^2 \mathbf{x}^H \mathbf{x},$$

where $\lambda_{\max}$ and $\lambda_{\min}$ are the maximum and minimum singular values of $\mathbf{H}$. Use the inequality above to show that

$$\min_{\mathbf{x}_i \neq \mathbf{x}_j} \left\| \mathbf{H}(\mathbf{x}_i - \mathbf{x}_j) \right\|_2^2 \geq \lambda_{\min}^2 \min_{\mathbf{x}_i \neq \mathbf{x}_j} \left\| (\mathbf{x}_i - \mathbf{x}_j) \right\|_2^2$$

3. (15%) Let the singular value decomposition (SVD) of an N x N full-rank matrix $\mathbf{H}$ be expressed as $\mathbf{H} = \mathbf{ABC}$.

(1) (10%) Detail the properties of matrices **A**, **B**, and **C**.

(2) (5%) Show that SVD is not unique if the diagonal elements of **B** are not necessary with decreasing order.

4. (25%) Consider a linear system given as

$$\mathbf{y} = \sum_{i=1}^{2} \mathbf{H}_i \mathbf{x}_i,$$

where $\mathbf{H}_1$ and $\mathbf{H}_2$ are M x N matrices with $M = 2N$.

(1) (15%) Demonstrate how to solve $\mathbf{x}_1$ and $\mathbf{x}_2$ by using the null spaces of $\mathbf{H}_1$ and $\mathbf{H}_2$.

(2) (10%) What possible problems we may encounter in (1).

5. (25%) Let **A** be an N x N matrix.

(1) (5%) Describe the null space of **A**.

(2) (10%) Show that the null space of **A** is a subspace.

(3) (5%) Describe the Rank Theorem.

(4) (5%) Use Rank Theorem to show that the columns of **A** are linearly dependent if **A** is not of full-rank.