1. Consider the differential equation $\dfrac{dy}{dx} = (x+y)(x+y-2)$ and one specific

   solution is given as $y = S(x) = a - x$.

   (a) (5%) Calculate the value of a.

   (b) (10%) Find the general solution of the equation.

2. (20%) Solve the differential equation $x^2 y'' - 2xy' + 2y = \cos(\dfrac{1}{x})$, $x > 0$.

3. (15%) Solve the system of linear differential equations.

   $2x_1 + 2x_2 + x_2' = \delta(t-3)$
   $x_1' - x_2 = 0$; $x_1(0^-) = 0$, $x_2(0^-) = 0$.

4. (15%) Consider the matrix $A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Calculate the eigenvectors of A.

5. Consider the space S spanned by two functions e^{-2x} and e^{2x}, using

   $p(x) = 1$ in the weighted inner product integral in $C[0, 1]$, i.e.,

   $f \cdot g = \int_0^1 p(x) f(x) g(x) dx$.

   (a) (10%) Find an orthogonal set of functions that spans the same subspace S.

   (b) (10%) Find the best approximation of $f(x) = x$ on $[0, 1]$ with a linear

   combination of the orthogonal basis found in (a).

6. Consider the matrix $A = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

   (a) (10%) Show that $\mathbf{u}$ can be obtained by rotating $\mathbf{v}$ about the z axis

   through angle α if $\mathbf{u} = A\mathbf{v}$.

   (b) (5%) Compute A^{20}.