注意事項：

1.本試題共 5 題，配分共 100 分。

2.請標明大題、子題編號作答，不必抄題。

3.全部答案均須在答案卷之答案欄內作答，否則不予計分。

1. Let $A = \begin{bmatrix} 0.5 & -0.5 & -0.5 & -0.5 \\ -0.5 & 0.5 & -0.5 & -0.5 \\ -0.5 & -0.5 & 0.5 & -0.5 \\ -0.5 & -0.5 & -0.5 & 0.5 \end{bmatrix}$

(1) Find the algebraic multiplicity of A (5%)

(2) Find the geometric multiplicity of A (5%)

(3) Compute A^n. (20%)

2. $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 5 \end{bmatrix}$

(1) Please find the Minimal Polynomial of A. (10%)

(2) Please find the nullity of $A - \lambda_i I$, where λ_i is the eigenvalue. (5%)

3. Let $A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 2 & -1 & 0 & 2 \\ -1 & 0 & 0 & 1 \\ 4 & 1 & -1 & 0 \end{bmatrix}$

Please utilize the **Cayley-Hamilton Theorem** to calculate A^{-1}. (15%)

4. Please prove the following theorems.

Let $A, B \in \mathbf{F}^{n \times m}$. Then,

(1) $\left. \begin{array}{c} \max\{rank(A),\ rank(B)\} \\ rank(A+B) \end{array} \right\} \leq \left\{ \begin{array}{c} rank\begin{bmatrix} A & B \end{bmatrix} \\ rank\begin{bmatrix} A \\ B \end{bmatrix} \end{array} \right\} \leq rank(A) + rank(B)$ (10%)

(2) $A \in \mathbf{F}^{n \times m}$, $B \in \mathbf{F}^{m \times l}$, and $C \in \mathbf{F}^{l \times k}$. Then,

$rank(AB) + rank(BC) \leq rank \begin{bmatrix} 0 & AB \\ BC & B \end{bmatrix} = rank(B) + rank(ABC)$ (10%)

5. Please prove the following theorems.

(1) Let $A \in \mathbf{F}^{n \times n}$. Then, for all $t \in \mathbf{R}$,

$e^{tA} - I = \int_0^t A e^{\tau A} d\tau$ and $\dfrac{d}{dt} e^{tA} = A e^{tA}$. (15%)

(2) Assume that all eigenvalues of $A \in \mathbf{F}^{n \times n}$ have negative real parts.

Then, for all $t \in \mathbf{R}$, $\int_0^\infty e^{tA} dt = -A^{-1}$. (5%)